An Asymptotic Method for Predicting Amplitudes of Nonlinear Wheel Shimmy

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An asymptotic method involving the multiple-time-scale perturbation technique is presented for nonlinear stability analysis of landing gear wheel shimmy models that include a velocity-squared damper. General expressions for the limit cycle amplitude and frequency are obtained, with the stability of the limit cycles determined by the sign of a computed coefficient. It is found that for positive values of this coefficient, a stable limit cycle exists for velocities exceeding a critical value and that complete stability ensues for velocities less than the critical velocity. For negative values of the coefficient, an unstable limit cycle exists for less-than-critical velocities, and the system is unstable for velocities exceeding the critical value. A simple shimmy model with a nonlinear damper is analyzed, and the limit cycle results obtained by the perturbation solution are shown to be in agreement with those calculated by numerical integration of the equations of motion.

Nomenclature

A = response amplitude
$[\bar{C}]$ = damping matrix
$\{\bar{d}\}$ = coefficient vector for nonlinear damping
$\{d\}$ = coefficient vector for nonlinear damping in state variable equations
$\{F^{(k)}\}$ = Fourier series coefficient
i = imaginary unit
K = constant dependent on initial conditions
$[\bar{K}]$ = stiffness matrix
$[K]$ = stiffness matrix in state variable equations
$[K^{(0)}]$ = zeroth-order matrix in expansion for $[K]$
$[K^{(1)}]$ = first-order matrix in expansion for $[K]$
$[L]$ = operator for zeroth-order system
$[\bar{M}]$ = mass matrix
$[M]$ = mass matrix in state variable equations
$\{q\}_m$ = coordinate vector in expansion for $\{x\}$
q_{im} = ith component of $\{q\}_m$
$\{u\}$ = right eigenvector
$\{u^*\}$ = complex conjugate of $\{u\}$
u_m = mth component of $\{u\}$
$\{v\}$ = left eigenvector
V = taxi velocity
V_0 = critical velocity
$\{\bar{x}\}$ = vector of dependent variables
$\{x\}$ = vector of state variables
$\bar{x}_m$ = mth component of $\{\bar{x}\}$
x_m = mth component of $\{x\}$
α = complex coefficient
β = complex coefficient
β_R = real part of β
β_I = imaginary part of β
γ = complex coefficient
γ_R = real part of γ
γ_I = imaginary part of γ
ϵ = small parameter
ϕ = part of ψ dependent on τ_1, τ_2, etc.
τ = time
τ_m = multiple time scales
ψ = phase angle
Ω = nonlinear system frequency
Ω_0 = eigenvalue

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Introduction

IN the design of new aircraft and the safe maintenance of operational ones, the prevention of landing gear shimmy continues to be an important area of concern. Wheel shimmy is fundamentally a self-excited oscillation caused by coupling of the lateral deflections (in a direction perpendicular to the taxi direction) and torsional oscillations about the gear swivel axis. The energy necessary to sustain the self-excitation is provided by the tire-ground reaction forces and moments exerted on the pneumatic tire. A linear or nonlinear shimmy analysis is generally performed and by appropriate changes in the geometric, damping, and structural parameters, a shimmy-free operation is sought. A linear description of the shimmy mechanism will generally fail to accurately predict the behavior of a landing gear system although it will usually reveal basic characteristics. This failure of linear analyses arises from the fact that such systems are inherently nonlinear due to nonlinear tire/ground reactions, Coulomb friction in the strut oleos, lateral and torsional freeplay in the torque links, and the use of velocity-squared hydraulic shimmy dampers. In many cases, as a flight test program proceeds, shimmy will suddenly occur for conditions which had previously been shimmy free. The cause is usually due either to tire wear or to increased tolerances in the gear fittings, which in some cases can be remedied by a change in tires and gear tightening. For severe cases, hydraulic shimmy dampers are added and a nonlinear analysis is required to specify the damper requirements and freeplay tolerances.

Few analytical solutions to nonlinear shimmy problems have appeared in the published literature. Pacejka[1] presented a detailed study of the nonlinear shimmy of automobiles including a nonlinear tire theory as well as investigations of the effects of dry friction in the kingpins and wheel bearing clearance on shimmy. The results identified limit cycles and indicated that linear tire theories produce substantially the same results as the nonlinear except in regions of high slippage. Collins[2] has also emphasized this latter point.

A standard solution method for nonlinear shimmy analysis is direct numerical integration of the equations of motion on a digital computer. Analog or hybrid computers are also used but can be extremely expensive due to the extensive setup time of the study. Digital computer costs can be equally expensive when time histories are obtained for a number of parametric variations.

A recent study by Nybakken[3] of linear shimmy models with parametric excitation considered the effects of tire variations on wheel shimmy for a simple aircraft nose gear model. A combination of Floquet theory and numerical integration was used to examine the stability of the resulting linear Hill-type equations. Podgorski[4] used linear models to examine the effects on wheel shimmy of a braking force, acceleration, oscillations of the vertical tire load and the forward velocity, wheel unbalance, out of roundness, and road roughness. He also used Floquet theory and numerical integration in the stability analysis.

It appears that aircraft shimmy analyses will continue to incorporate linear tire mechanics theories with nonlinearities due to Coulomb friction, freeplay, and velocity-squared shimmy dampers in order to accurately predict the behavior of these systems. Thus, additional analytical solution methods to facilitate analysis of these nonlinearities warrant consideration. In this study, an analytical procedure for nonlinear shimmy analyses including a velocity-squared damper is proposed as an alternative to conventional shimmy analysis.

The nonlinear analysis for the self-excited system is based on the multiple-time-scale technique.[5] Previously, applications to similar systems have been made by Morino[6] and Kuo et al.[7] for the flutter of panels, Smith and Morino[8] for the flutter of panels and wings, and by Gordon and Atluri[9] for the flutter of cylindrical shells. In these aeroelastic applications the nonlinearities have been either quadratic or cubic in the dependent variables when expressed in state variable form.

Problem Formulation and Solution

For a lumped mass model of a landing gear system with a velocity squared shimmy damper, the dynamic equations of motion can be written as

$$[\bar{M}]\{\ddot{x}\}+[\bar{C}]\{\dot{x}\}+[\bar{K}]\{x\}+\text{sgn}(\dot{x}_m)\dot{x}_m{}^2\{\bar{d}\}=\{0\} \tag{1}$$

where the inertia, damping, and stiffness matrices, $[\bar{M}]$, $[\bar{C}]$, and $[\bar{K}]$, are, in general, functions of the taxi velocity V. The term $\text{sgn}\,(\dot{x}_m)\dot{x}_m{}^2\{\bar{d}\}$ is due to the nonlinear shimmy damper where $\{\bar{d}\}$ is a coefficient matrix for nonlinear damping.

The equations in state variables are

$$[M]\{\dot{x}\}+[K]\{x\}+\text{sgn}(x_m)x_m{}^2\{d\}=0 \tag{2}$$

where for simplicity, assume that

$$[K]=V[K^{(1)}]+[K^{(0)}] \tag{3}$$

and

$$\text{sgn}(x_m)=(+1,0,-1)\quad\text{for}\quad(x_m>0,=0,<0) \tag{4}$$

There is a critical value $V=V_0$ such that the linear system ($\{d\}=0$) is stable for $V<V_0$ and unstable for $V>V_0$. For $V=V_0$, one pair of eigenvalues are pure imaginary and all other eigenvalues have negative real parts. Thus, undamped harmonic oscillations occur for $V=V_0$. The solution of Eq. (2) is sought in the neighborhood of the critical point $V=V_0$.

Assume that there exists a uniformly valid asymptotic expansion for the dependent variables $\{x\}$ of the form

$$\{x\}=\epsilon\sum_{m=0}^{M}\epsilon^m\{q\}_m \tag{5}$$

where ϵ is a small parameter. Define multiple time scales

$$\tau_m=\epsilon^m\tau \qquad (m=0,1,2...)$$

Therefore

$$\frac{\mathrm{d}}{\mathrm{d}\tau}=\frac{\partial}{\partial\tau_0}+\frac{\epsilon\partial}{\partial\tau_1}+0(\epsilon^2) \tag{6}$$

Assume that V can be expanded about V_0 as

$$V=V_0+\epsilon V_1+0(\epsilon^2) \tag{7}$$

The term $\text{sgn}\,(x_m)^2x_m{}^2$ can be expressed as

$$\text{sgn}(x_m)x_m{}^2=\epsilon^2[\text{sgn}(q_{m0})q^2{}_{m0}+2\epsilon|q_{m0}|q_{m1}]+0(\epsilon^4) \tag{8}$$

where

$$\{q\}^T{}_m=\lfloor q_{1m},q_{2m},...\rfloor \tag{9}$$

Substituting Eqs. (5-9) into Eq. (2) and equating terms in powers of ϵ gives the following equations:

$$\epsilon^0:\qquad [L]\{q\}_0=\{0\} \tag{10}$$

$$\epsilon^1:\qquad [L]\{q\}_1=-\text{sgn}(q_{m0})q_{m0}{}^2\{d\}-[M]\partial\{q\}_0/\partial\tau_1 - V_1[K^{(1)}]\{q\}_0 \tag{11}$$

where

$$[L]=[M]\partial/\partial\tau_0+V_0[K^{(1)}]+[K^{(0)}] \tag{12}$$

The system represented by Eq. (10) defines the linear shimmy problem. At some critical value of the parameter $V=V_0$ a harmonic solution is obtained for $\{q\}_0$, namely,

$$\{q\}_0=2Re[A\{u\}e^{i\psi}] \tag{13}$$

where

$$\psi=\Omega_0\tau_0+\phi(\tau_1,\tau_2,...)$$

and

$$A=A(\tau_1,\tau_2,...)$$

is real. Ω_0 and $\{u\}$ are the eigenvalue and eigenvector defined by the equations

$$[i\Omega_0[M]+V_0[K^{(1)}]+[K^{(0)}]]\{u\}=\{0\} \tag{14}$$

Substituting $\{q\}_0$ from Eq. (13) into Eq. (11) and expanding the term $\{d\}\text{sgn}\,(q_{m0})\,q_{m0}{}^2$ in a complex Fourier series yields

$$[L]\{q\}_1=-e^{i\psi}\left[A^2\{F^{(1)}\}+\left(\frac{\partial A}{\partial\tau_1}+iA\frac{\partial\psi}{\partial\tau_1}\right)[M]\{u\} + V_1[K^{(1)}]\{u\}A\right]-e^{-i\psi}\left[A^2\{F^{(-1)}\}+\left(\frac{\partial A}{\partial\tau_1}-iA\frac{\partial\psi}{\partial\tau_1}\right)[M]\{u\} + V_1[K^{(1)}]\{u^*\}A\right]-\sum_{\substack{-\infty<k<\infty\\k\neq\pm1}}A^2\{F^{(k)}\}e^{ik\psi} \tag{15}$$

where

$$A^2\{F^{(k)}\}=\frac{1}{2\pi}\int_0^{2\pi}\{d\}\text{sgn}(q_{m0})q_{m0}{}^2e^{-ik\psi}\mathrm{d}\psi \tag{16}$$

$\{u^*\}$ is the complex conjugate of $\{u\}$, and

$$q_{m0} = A(u_m e^{i\psi} + u_m^* e^{-i\psi})$$

where q_{m0} and u_m are the mth components of $\{q\}_0$ and $\{u\}$, respectively.

It can be shown that the terms on the right side of Eq. (15), which multiply exp $(i\psi)$ and exp $(-i\psi)$, lead to spurious resonance unless the following condition is satisfied[10]:

$$\lfloor v \rfloor \{A^2\{F^{(I)}\} + \left(\frac{\partial A}{\partial \tau_I} + i\frac{\partial \psi}{\partial \tau_I}A\right)[M]\{u\} + V_I[K^{(I)}]\{u\}A\} = 0 \quad (17)$$

where $\{v\}$ is the left eigenvector associated with

$$\lfloor v \rfloor [i\Omega_0[M] + V_0[K^{(I)}] + [K^{(0)}]] = \lfloor 0 \rfloor \quad (18)$$

Equation (17) leads to differential equations for the amplitude A and phase ψ, namely,

$$\frac{\partial A}{\partial \tau_I} + \beta_R A + \gamma_R A^2 = 0 \quad (19)$$

$$\frac{\partial \psi}{\partial \tau_I} + \beta_I + \gamma_I A = 0 \quad (20)$$

where

$$\beta_R + i\beta_I = V_I \lfloor v \rfloor [K^{(I)}]\{u\}/\alpha \quad (21)$$

$$\gamma_R + i\gamma_I = \lfloor v \rfloor \{F^{(I)}\}/\alpha \quad (22)$$

and

$$\alpha = \lfloor v \rfloor [M]\{u\} \quad (23)$$

The solution to Eq. (19) is

$$A = -\beta_R/(Ke^{\beta_R \tau_I} + \gamma_R) \quad (24)$$

where $K = K(\tau_2, \tau_3, \ldots)$. The solution to Eq. (20) is

$$\psi = \Omega_0 \tau_0 + \left(\frac{\beta_R \gamma_I}{\gamma_R} - \beta_I\right)\tau_I + \phi(\tau_2, \ldots) \quad (25)$$

where

$$\phi = \frac{-\gamma_I}{\gamma_R} \ln(Ke^{\beta_R \tau_I} + \gamma_R) + \phi_0(\tau_2, \ldots) \quad (26)$$

Limit Amplitude Solution

From Eq. (24) when $\gamma_R > 0$

$$\lim_{\substack{\tau_I \to \infty \\ \gamma_R > 0}} A = \begin{cases} -\beta_R/\gamma_R & \text{if } V > V_0 \\ 0 & \text{if } V < V_0 \end{cases} \quad (27)$$

since for $V > V_0$ the linear system ($\gamma = 0$) is unstable ($\beta_R < 0$), and for $V < V_0$ the linear system is stable ($\beta_R > 0$).

When $\gamma_R < 0$ and $V > V_0$, i.e., $\beta_R < 0$

$$\lim_{\tau_I \to \tau_{I0}} A = \infty \quad (28)$$

where $\tau_{I0} = -(1/\beta_R)\ln|K/\gamma_R|$. When $\gamma_R < 0$ and $V < V_0$, i.e. $\beta_R > 0$,

$$\lim_{\tau_I \to -\infty} A = -\beta_R/\gamma_R \quad (29)$$

and as time increases

$$\lim_{\tau_I \to \tau_{I0}} A = \infty \text{ if } A|_{\tau_I = 0} > -\beta_R/\gamma_R \quad (K > 0) \quad (30a)$$

$$\lim_{\tau_I \to \infty} A = 0 \text{ if } A|_{\tau_I = 0} < -\beta_R/\gamma_R \quad (K < 0) \quad (30b)$$

The complete solution for $\{x\}$ is obtained by combining Eqs. (5, 7, 13, and 24-26) as,

$$\{x\} = 2\epsilon Re[A\{u\}e^{i\psi}] + O(\epsilon^2) \quad (31)$$

where ϵ, A, and ψ are given by Eqs. (7, and 24-26) where $\epsilon = (V - V_0)/V_I + O(\epsilon^2)$.

Thus, as $\tau_0 \to \infty$, a limit cycle exists of the form

$$\{x\} = 2\left|\frac{(V - V_0)}{V_I}\left(\frac{\beta_R}{\gamma_R}\right)\right| Re[\{u\}e^{i(\Omega\tau_0 + \phi)}] + O(\epsilon^2) \quad (32)$$

where

$$\Omega = \Omega_0 + \epsilon(\beta_R \gamma_I/\gamma_R - \beta_I) \quad (33)$$

$$\phi = -(\gamma_I/\gamma_R)\ln(Ke^{\beta_R \tau_I} + \gamma_R) + \phi_0(\tau_2, \ldots) \quad (34)$$

and ϕ is a constant. For $\gamma_R > 0$, a stable limit cycle exists for $V > V_0$ and the system is stable for $V < V_0$ as shown in Fig. 1.

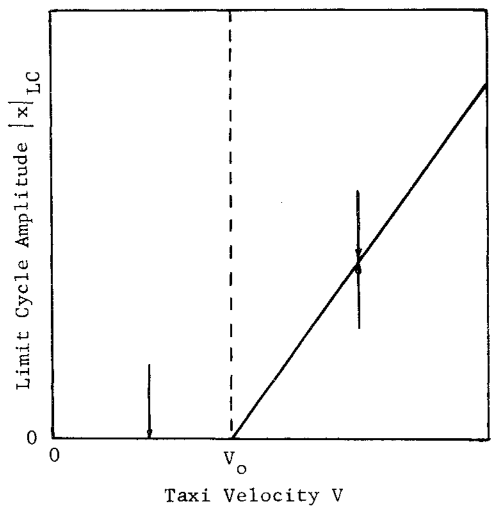

Fig. 1 Limit cycle amplitude $|x|_{LC}$ vs taxi velocity V for $\gamma_R > 0$.

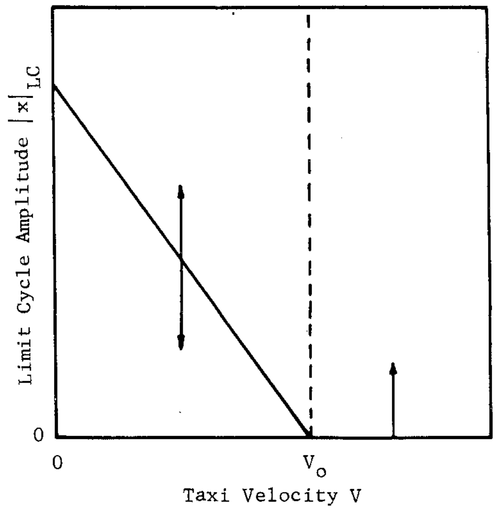

Fig. 2 Limit cycle amplitude $|x|_{LC}$ vs taxi velocity V for $\gamma_R < 0$.

For $\gamma_R < 0$, an unstable limit cycle exists for $V < V_0$ and the system is unstable for $V > V_0$ as shown in Fig. 2.

Numerical Example

As an application of the preceding analysis, a simple shimmy model with a nonlinear damper is examined. The model shown in Fig. 3 is a modification of one studied by Nybakken[3] consisting of a wheel attached to a pivoted rigid arm. The system has a moment of inertia I about the pivot, a mechanical trail L_m, a pivotal stiffness k_t, a parallel linear damper C_t, and a parallel nonlinear (velocity-squared) damper C_d. A finite contact patch is assumed and the tangent approximation to Von Schlippe's linear string theory (Nybakken)[3] is used for the tire model. The equations of motion in state variables from Eq. (2) are

$$\begin{bmatrix} I & C_t & 0 \\ 0 & 0 & \lambda \\ 0 & 1 & 0 \end{bmatrix} \{\dot{x}\} + \begin{bmatrix} 0 & k_1 & k_3 \\ 0 & k_4 V & V \\ -1 & 0 & 0 \end{bmatrix} \{x\}$$

$$+ \begin{Bmatrix} C_d \\ 0 \\ 0 \end{Bmatrix} x_1{}^2 \mathrm{sgn}(x_1) = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \tag{35}$$

where

$$\{x\}^T = \lfloor \dot{\theta}\theta y \rfloor = \lfloor x_1, x_2, x_3 \rfloor \tag{36}$$

and

$$k_1 = k_t + k_2 \tag{37}$$

$$k_2 = k_y \bar{L}(1 + h/\lambda)(L_m - h) \tag{38}$$

$$k_3 = -k_y \bar{L}(1 + h/\lambda) \tag{39}$$

$$k_4 = -L_m + \lambda + h \tag{40}$$

$$\bar{L} = L_m + L_p \tag{41}$$

and

I = 189 lb-s^2-in. = moment of inertia about pivot
C_t = 2268 lb-s-in. = pivot linear damping coefficient
λ = 10.24 in. = tire relaxation length
k_t = 2.4741 (10^6) lb-in./rad = pivot stiffness
k_y = 6861.4 lb/in. = tire lateral stiffness
L_m = 3.0 in. = mechanical trail of wheel
L_p = 5.75 in. = pneumatic trail of tire
h = 6.72 in. = half-length of the tire contact patch
C_d = 2268 lb-s^2-in. = nonlinear damping coefficient
V = forward velocity
θ = rotation of wheel about the pivot
y = lateral coordinate of the front of the contact patch from reference line

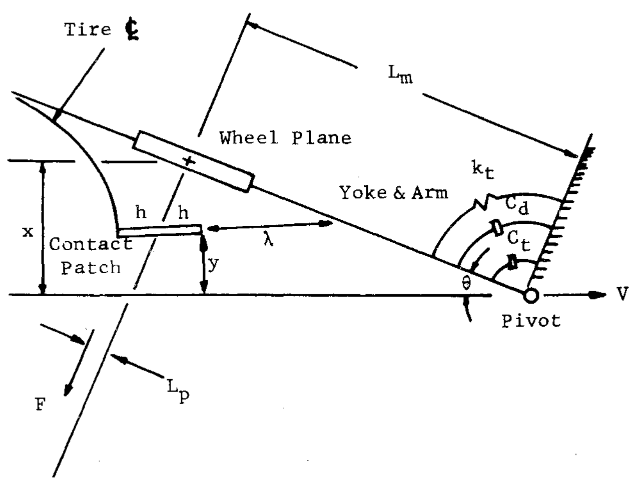

Fig. 3 Nonlinear shimmy model.

These numerical values are typical of a large commercial aircraft nose gear.

For the linear system (ϵ^0 order), Eq. (14) gives the shimmy speed V_0 and frequency Ω_0.

$$V_0{}^2 + V_0\left(\frac{\lambda C_t}{I} + \frac{k_3 k_4 \lambda}{C_t}\right) + \frac{k_1 \lambda^2}{I} = 0 \tag{42}$$

$$\Omega_0{}^2 = (\lambda k_1 + C_t V_0)/(I\lambda) \tag{43}$$

The right and left eigenvectors $\{u\}$ and $\{v\}$ corresponding to Ω_0 and V_0 are

$$\{u\} = \begin{Bmatrix} i\Omega_0 \\ 1 \\ -k_4 V_0/(V_0 + i\lambda\Omega_0) \end{Bmatrix} \tag{44}$$

$$\{v\} = \begin{Bmatrix} 1 \\ -k_3/(i\lambda\Omega_0 + V_0) \\ iI\Omega_0 \end{Bmatrix} \tag{45}$$

α, β, and γ are computed using Eqs. (21-23) as

$$\alpha_R = C_t + \frac{\lambda k_3 k_4 V_0 (V_0{}^2 - \lambda\Omega_0{}^2)}{\lambda^2 \Omega_0{}^2 + V_0{}^2} \tag{46}$$

$$\alpha_I = 2I\Omega_0 - \frac{2\lambda^2 k_3 k_4 V_0{}^2 \Omega_0}{\lambda^2 \Omega_0{}^2 + V_0{}^2} \tag{47}$$

$$\alpha = \alpha_R + i\alpha_I$$

$$\beta = \frac{V_I k_3 k_4}{\alpha}\left[\frac{-2\lambda^2\Omega_0{}^2 V_0 + i(\lambda^3\Omega_0{}^3 - \lambda\Omega_0 V_0{}^2)}{(V_0{}^2 + \lambda^2\Omega_0{}^2)}\right] \tag{48}$$

$$\gamma = \frac{16\Omega_0{}^2 C_d i}{3\pi\alpha} \tag{49}$$

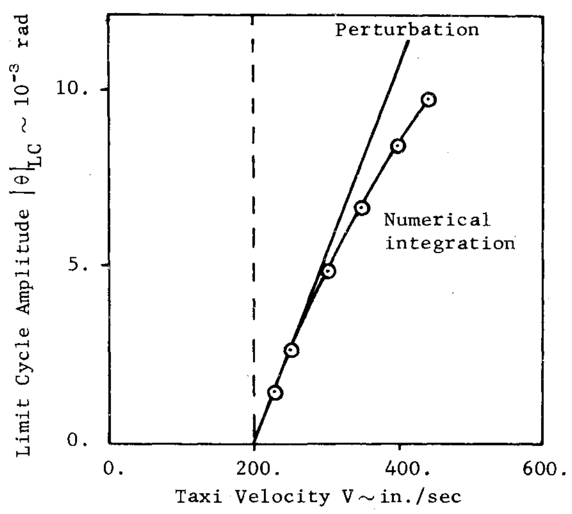

Fig. 4 Limit cycle amplitude $|\theta|_{LC}$ vs taxi velocity V.

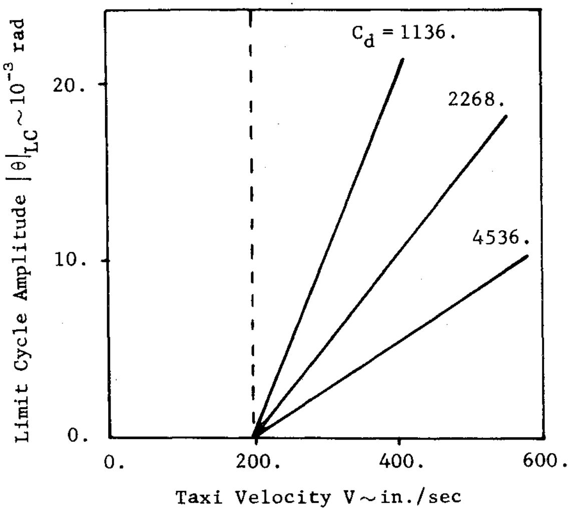

Fig. 5 Limit cycle amplitude $|\theta|_{LC}$ vs taxi velocity V as a function of nonlinear damping.

and the limit cycle amplitude is given by Eq. (32). Note that the limit cycle amplitude is inversely proportional to the nonlinear damping coefficient C_d and proportional to the taxi velocity V. Figure 4 presents curves of limit cycle amplitude θ vs velocity for the preceding numerical values. Figure 4 also presents a comparison of the perturbation solution with results from direct numerical integration of the equations. Figure 5 shows the effect of a change in the nonlinear damping coefficient C_d.

Conclusions

A nonlinear analysis has been presented for wheel shimmy models which include a velocity-squared damper. General expressions for the limit cycle amplitude and frequency have been obtained with the stability of the limit cycles determined by the sign of a computed coefficient γ_R. For $\gamma_R > 0$, a stable limit cycle has been shown to exist for $V > V_0$ with complete stability for $V < V_0$; for $\gamma_R < 0$, an unstable limit cycle exists for $V < V_0$ and the system is unstable for $V > V_0$. A simple shimmy model has been investigated and the limit cycle results obtained by the perturbation solution have been shown to be in good agreement with those from numerical integration of the equations of motion.

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